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GLLAMM models with discrete latent variables

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Outline

- General introduction to GLLAMM
- Exploratory and confirmatory latent class models
- Examples
  - Binary items on attitudes to abortion from a cross-sectional survey of complex design
  - Binary/ordinal diagnoses of autism from a longitudinal study
  - Ranked responses to items from a cross-sectional study of post-materialism

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Generalized Linear Latent and Mixed Models: GLLAMM

Response model: linear predictor

(Illustrated for two level model)

Conditional on the latent variables, the response model is a generalized linear model with linear predictor

$$\nu_{ij} \;=\; \mathbf{x}'_{ij}\boldsymbol{\beta} + \sum_{m=1}^{M} \eta_{jm}\mathbf{z}'_{ij}\boldsymbol{\lambda},$$

- i indexes the units at level 1 (variables or items)
- j indexes the units at level 2 (units or subjects), with $i = 1, \cdots, n_j$
- $\boldsymbol{\beta}$ and $\boldsymbol{\lambda}$ are parameters,
- $\mathbf{x}_{ij}$ and $\mathbf{z}_{ij}$ are vectors of observed variables and known constants
- η_{jm} is the mth element of the latent variable vector $\boldsymbol{\eta}_j$.

Latent variables can be correlated, and can be dependent and/or independent variables

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Response Model: The conditional distribution

- Conditional expectation:

  $$\mathrm{E}[y_{ij}|\mathbf{x}_{ij}, \mathbf{z}_{ij}, \boldsymbol{\eta}_j] \;=\; g^{-1}(\nu_{ij}),$$

  where $g(\cdot)$ is the link function
- The conditional distributions of the responses are from the exponential family
- The responses are conditionally independent given the latent variables
- The responses can be of mixed types (different links and distributions)

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Response Model: Response types

- Continuous (normal, gamma)
- Binary (logit, probit or complementary log-log links)
- Ordinal
  - Cumulative models (logit, probit or complementary log-log links)
    including models for thresholds or scale parameter
  - Adjacent category odds model
  - Continuation ratio model
- Unordered categorical and rankings (multinomial logit)
- Counts (Poisson, binomial)
- Durations in continuous time
  - Proportional hazards model
  - Accelerated failure time model
- Durations in discrete time
  - Censored cumulative models
  - Continuation ratio model
  - Proportional hazards in continuous time
- Mixed responses

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`gllamm` Platform, Command and Data Structure

- `gllamm` is implemented as a set of Stata procedures (MS Windows/NT, Mac and Unix/Linux)
- `gllamm` command for point and precision estimation
- `gllapred` command for prediction including empirical Bayes
- `gllasim` command for simulation given model, explanatory variables and parameter values
- "One response per record" data structure (MAR for missing responses under ML - no balance assumed)

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Discrete Latent Variables

- Latent variable vector $\boldsymbol{\eta}_j$ for unit j with discrete values (or locations) $\boldsymbol{e}_c, c\!=\!1, \cdots, C$ in M dimensions.
- Individuals in the same latent class share the same value or location $\boldsymbol{e}_c$.
- Let π_{jc} denote the (prior) probability that unit j is in latent class c.
- This probability may depend on covariates $\mathbf{v}_j$ through a multinomial logit model

$$\pi_{jc} \;=\; \frac{\exp(\mathbf{v}_j'\boldsymbol{\alpha}_c)}{\sum_d \exp(\mathbf{v}_j'\boldsymbol{\alpha}_d)},$$

  where $\boldsymbol{\alpha}_c$ are parameters with $\boldsymbol{\alpha}_1\!=\!0$ for identification.

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Model Likelihood

Joint distribution of responses $\boldsymbol{y}_j$ for unit j:

$$\Pr(\boldsymbol{y}_j|\mathbf{X}_j, \boldsymbol{Z}_j, \mathbf{v}_j; \boldsymbol{\theta}) \;=\;$$

$$\sum_{c=1}^{C} \Pr(\boldsymbol{\eta}_j\!=\!\boldsymbol{e}_c|\mathbf{v}_j; \boldsymbol{\alpha}) \prod_{i=1}^{n_j} f(y_{ij}|\mathbf{X}_j, \boldsymbol{Z}_j, \boldsymbol{\eta}_j\!=\!\boldsymbol{e}_c; \boldsymbol{\beta}, \boldsymbol{\lambda})$$

- $\mathbf{X}_j$ and $\boldsymbol{Z}_j$ are $n_j \times p$ and $n_j \times q$ design matrices for the vector of responses,
- $\boldsymbol{\theta}$ is the vector of all parameters
- $f(y_{ij}|\mathbf{X}_j, \boldsymbol{Z}_j, \boldsymbol{\eta}_j\!=\!\boldsymbol{e}_c; \boldsymbol{\beta}, \boldsymbol{\lambda})$ is the conditional probability (density) of the response given the observed and latent variables.

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Conventional Exploratory Models

Conventional exploratory latent class model imposes no structure on the conditional response probabilities. The linear predictor has the form

$$\nu_{ijc} \;=\; \beta_i + e_{ic}.$$

The model can be written as

$$\nu_{ijc} \;=\; \boldsymbol{d}_i'\boldsymbol{\beta} + \sum_{m=1}^{I} e_{mc} d_{mi},$$

where $\boldsymbol{d}_i$ is a vector of length I with ith element equal to 1 and all other elements equal to 0, $\boldsymbol{d}_i = (d_{1i}, \cdots, d_{Ii})'$ where

$$d_{mi} \;=\; \begin{cases} 1 & \text{if } m = i \\ 0 & \text{if } m \neq i \end{cases}$$

Here e_{mc} is the cth location of the mth latent variable.

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"Confirmatory" Model

- One-factor model would be specified by

$$\nu_{ijc} \;=\; \beta_i + \lambda_i e_c.$$

- In GLLAMM framework the model can be written as

$$\nu_{ijc} \;=\; \boldsymbol{d}_i'\boldsymbol{\beta} + e_c \boldsymbol{d}_i'\boldsymbol{\lambda}.$$

- If we increase the number of classes until the likelihood cannot be increased any further, the discrete distribtution can be viewed as a nonparametric maximum likelihood (NPML) estimator of a continuous distribution.

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Example 1 : Attitudes to abortion data

- British Social Attitudes Survey 1983
- Respondents were asked whether or not abortion should be allowed by law if:

[wom] The woman decides on her own she does not wish to have the child

[cou] The couple agree that they do not wish to have the child

[mar] The woman is not married and does not wish to marry the man

[fin] The couple cannot afford any more children

[gen] There is a strong chance of a genetic defect in the baby

[ris] The woman's health is seriously endangered by the pregnancy

[rap] The woman became pregnant as a result of rape

- 720 respondents, 89% have complete data, a total of 7% of items are missing

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Attitudes to abortion: Data structure

id	ab	wom	cou	mar	fin	gen	ris	rap	fem	wt2	pwt2	area83
1	1	1	0	0	0	0	0	0	0	1	.8281	102
1	1	0	1	0	0	0	0	0	0	1	.8281	102
1	1	0	0	1	0	0	0	0	0	1	.8281	102
1	1	0	0	0	1	0	0	0	0	1	.8281	102
1	1	0	0	0	0	1	0	0	0	1	.8281	102
1	1	0	0	0	0	0	1	0	0	1	.8281	102
1	1	0	0	0	0	0	0	1	0	1	.8281	102
2	0	1	0	0	0	0	0	0	0	1	.621075	102
2	1	0	1	0	0	0	0	0	0	1	.621075	102
2	1	0	0	1	0	0	0	0	0	1	.621075	102
2	1	0	0	0	1	0	0	0	0	1	.621075	102
2	1	0	0	0	0	1	0	0	0	1	.621075	102
2	1	0	0	0	0	0	1	0	0	1	.621075	102
2	1	0	0	0	0	0	0	1	0	1	.621075	102
3	1	1	0	0	0	0	0	0	0	1	.8281	102
3	1	0	1	0	0	0	0	0	0	1	.8281	102
...												

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Attitudes to abortion: Model specification

- Model 1: Discrete one-factor two-class model

$$\text{logit}[\Pr(y_{ij}=1|\eta_j=e_c)] = \beta_i + \lambda_i e_c,$$

```
eq fac: wom cou mar fin gen ris rap
gllamm ab wom cou mar fin gen ris rap, nocons
    weight(wt) i(id) l(logit) f(binom)
    eqs(fac) ip(f) nip(2)
```

- Model 2: Class probabilities depend on sex (v_j=`[fem]`)

$$\pi_{j1} = \frac{\exp(\alpha_0+\alpha_1 v_j)}{1+\exp(\alpha_0+\alpha_1 v_j)},\ \pi_{j2} = 1-\pi_{j1}.$$

```
eq fem: fem
gllamm ab wom cou mar fin gen ris rap, nocons
     weight(wt) i(id) l(logit) f(binom)
     eqs(fac) peqs(fem) ip(f) nip(2)
```

- Model 3: Include a direct effect of gender on the second item `[cou]`.

$$\text{logit}[\Pr(y_{2j}=1|\eta_j=e_c, v_j)] = \beta_{02} + \beta_{12} v_j + \lambda_i e_c.$$

```
gen femcou = fem*cou
gllamm ab wom cou femcou mar fin gen ris rap, ...
```

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Parameter estimates for models with two latent classes

	Model 1	Model 2
Intercepts:		
β_1 [wom]	-0.49 (0.12)	-0.46 (0.12)
β_2 [cou]	0.39 (0.24)	0.60 (0.28)
β_3 [mar]	-0.19 (0.15)	0.06 (0.17)
β_4 [fin]	0.22 (0.14)	0.43 (0.16)
β_5 [gen]	2.69 (0.26)	2.86 (0.29)
β_6 [ris]	3.48 (0.47)	3.66 (0.52)
β_7 [rap]	2.85 (0.22)	2.95 (0.24)
Factor loadings:		
λ_1 [wom]	1 (–)	1 (–)
λ_2 [cou]	1.62 (0.24)	1.64 (0.24)
λ_3 [mar]	1.33 (0.16)	1.32 (0.16)
λ_4 [fin]	1.16 (0.15)	1.15 (0.15)
λ_5 [gen]	0.94 (0.22)	0.93 (0.21)
λ_6 [ris]	1.05 (0.39)	1.04 (0.38)
λ_7 [rap]	0.61 (0.19)	0.60 (0.18)
Locations parameter:		
e_1	-1.28 (0.14)	-1.47 (0.16)
Probability parameters (class 1):		
α_0 [cons]	0.24 (0.12)	-0.01 (0.17)
α_1 [fem]	–	0.48 (0.17)
Log-likelihood:	-1967.89	-1963.82

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Models Using Complex Survey Data

- British Attitudes Survey not a simple random sample
- Pseudolikelihoood estimation with inverse probability weights
- Robust standard errors (sandwich estimator) for cluster sampling with electoral ward as psu.

`gllamm` options `pweight` and `robust`:

```
gllamm ab wom cou mar fin gen ris rap, nocons
   weight(wt) i(id)l(logit) f(binom)
   eqs(fac) ip(f) nip(2) peqs(fem)
   pweight(pwt) robust cluster(area83)
```

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Design based adjustment for clustering and weighting using PML and sandwich variance estimator

	no weights model-based se	weights robust se	weights robust se, cluster
Intercepts:			
β_1 [wom]	-0.46 (0.12)	-0.26 (0.15)	(0.16)
β_2 [cou]	0.60 (0.28)	0.82 (0.39)	(0.40)
β_3 [mar]	0.06 (0.17)	0.04 (0.21)	(0.24)
β_4 [fin]	0.43 (0.16)	0.35 (0.18)	(0.21)
β_5 [gen]	2.86 (0.29)	2.81 (0.31)	(0.30)
β_6 [ris]	3.66 (0.52)	3.72 (0.58)	(0.61)
β_7 [rap]	2.95 (0.24)	2.87 (0.31)	(0.32)
Factor loadings:			
λ_1 [wom]	1 (-)	1 (-)	
λ_2 [cou]	1.64 (0.24)	1.67 (0.29)	(0.30)
λ_3 [mar]	1.32 (0.16)	1.31 (0.18)	(0.21)
λ_4 [fin]	1.15 (0.15)	1.12 (0.18)	(0.19)
λ_5 [gen]	0.93 (0.21)	0.87 (0.24)	(0.27)
λ_6 [ris]	1.04 (0.38)	1.12 (0.46)	(0.45)
λ_7 [rap]	0.60 (0.18)	0.57 (0.26)	(0.25)
Location parameter:			
e_1	-1.47 (0.16)	-1.40 (0.21)	(0.20)
Probability parameters (class 1):			
α_0 [cons]	-0.01 (0.17)	0.07 (0.21)	(0.19)
α_1 [fem]	0.48 (0.17)	0.43 (0.18)	(0.19)

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Attitudes to Abortion: Prediction using `gllapred`

```
* e1, e2, e3 are locations
gllapred mup, mu us(e)
gllapred mu, mu marg
```

Illustrated for model extended to 3 classes

	class 1	class 2	class 3		
Prior Probabilities					
male	6	47	47		
female	4	60	36		
Conditional Probabilities				Marginal	
				male	female
[wom]	0	18	78	45	38
[cou]	0	20	98	56	47
[mar]	0	16	91	51	42
[fin]	0	26	92	56	48
[gen]	7	90	98	89	89
[ris]	33	96	99	94	94
[rap]	53	93	97	93	93

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Example 2: Early Diagnosis of Autism

- Approx 130 children suspected of autism diagnosed by
  - structured interview [`adi`]
  - structured observation [`adosg`]
  - clinical judgement [`clin`]
- Children diagnosed as autistic, pdd-nos or non-autistic spectrum
- Assessments made at age 2,3,5 and 9
- Comparatively sparse data often leads to boundary or near boundary solutions.

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Two-class Model Fitted to Binary Age-2 Data (autism versus non-autism)

The simple two-class exploratory model for diagnosis measure i on child j

$$\text{logit}[\Pr(y_{ij} = 1|c] = \beta_i + e_{ic}.$$

is equivalent to the one-factor model

$$\text{logit}[\Pr(y_{ij} = 1|c] = \beta_i + \lambda_i e_c.$$

Intercepts	
β_1 interview	0.08 (0.24)
β_2 observation	6.68 (295)
β_3 clinician	-1.95 (0.61)
Conventional parameterisation:	
Locations (class 1)	
e_{11} interview	-1.07 (0.27)
e_{21} observation	-7.47 (295.)
e_{31} clinician	-1.98 (0.52)
Alternative parameterisation:	
Location (class 1)	
e_1	-1.07 (0.27)
Factor Loadings	
λ_2	6.41 (170.)
λ_3	1.84 (0.61)
Probability parameter (class 1)	
α_0	0.36 (0.24)

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Avoiding Boundary Solutions Using a Simulated Data Prior

For example, could
(i) estimate model with sensitivity and specificity of interview and observation set equal,
(ii) use `gllasim` to simulate data from this model,
(iii) append simulated data to real data and re-estimate with simulated data given suitable low weight e.g. here 4 replicates each weighted 0.025

Intercepts	
β_1 interview	0.11 (0.23)
β_2 observation	5.73 (34.7)
β_3 clinician	-1.89 (0.52)
One-factor parameterisation:	
Location (class 1)	
e_1	-1.07 (0.26)
Factor Loadings	
λ_2	6.07 (31.0)
λ_3	1.78 (0.55)
Probability parameter (class 1)	
α_0	0.33 (0.25)

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Two-factor Models: Latent Transition Model Age 2 to Age 5

- The model is defined by specifying classes for latent histories 00,01,10 and 11.
- We restrict the factor loadings to be the same at age 2 and age 5.

```
eq load2: adi_t2 adosg_t2 clin_t2
eq load5: adi_t5 adosg_t5 clin_t5
cons def 1 [id1_1l]adosg_t2 = [id1_2l]adosg_t5
cons def 2 [id1_1l]clin_t2 = [id1_2l]clin_t5
```

- We restrict the location of 0 at age 2 in the (00) class to be the same as that in the (01) class, and similarly for 1 age 2 in (10) and (11), 0 at age 5 in (01) and (11), and 1 at age 5 in (01) and (11).

```
cons def 3 [z2_1_1]adi_t2 = [z2_1_4]adi_t2
cons def 4 [z2_2_1]adi_t5 = [z2_2_2]adi_t5
cons def 5 [z2_1_2]adi_t2 = [z2_1_3]adi_t2
cons def 6 [z2_2_3]adi_t5 = [z2_2_4]adi_t5
```

- Fit the model with the constraints

```
gllamm aut, nocons i(id) nrf(2) eqs(load2 load5)
    ip(fn) nip(4) f(binom) constr(1 2 3 4 5 6)
```

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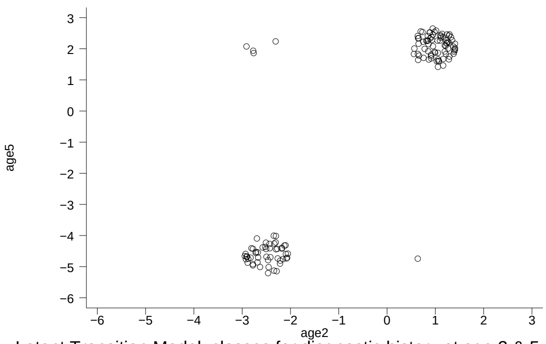
Autism Diagnosis: Child Classification to Latent Histories

- use

  ```
  gllapred p, p
  ```

  to calculate posterior class membership probabilities for each child, assigning each child to class with highest probability

Scatterplot of child class assignments (jittered)



Latent Transition Model: classes for diagnostic history at age 2 & 5

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Random coefficient models: Latent Trajectory Models

- Discrete random intercept and slope of time
- Factor loadings for the three instruments
- Use the 3-category diagnosis as an ordinal response

$$\ln\left(\frac{\Pr(y_{ij}>s)}{\Pr(y_{ij}\leq s)}\right) = \beta_i + \lambda_i(e_{1c} + e_{2c}a_t) - \kappa_s,$$

$$\beta_1 = \kappa_1 = 0, \ \lambda_1 = 1, \ s = 1, 2.$$

```
gen adi_t = adi*t
gen adosg_t = adosg*t
gen clin_t = clin*t
eq int: adi adosg clin
eq slope: adi_t adosg_t clin_t
```

- Set factor loadings equal for intercept and slope and set $\kappa_1 = 0$

```
cons def 1 [id1_1l]adosg = [id1_2l]adosg_t
cons def 2 [id1_1l]clin = [id1_2l]clin_t
cons def 3 [_cut11]_cons=0
```

- Estimate model

```
gllamm dx adosg clin, i(id) f(binom) nrf(4)
    eqs(int slope) ip(fn) nip(4) constr(1 2 3)
```

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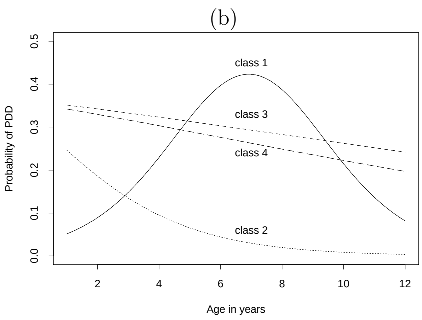

Conditional predicted probabilities of (a) autism and (b) pdd for the four latent classes

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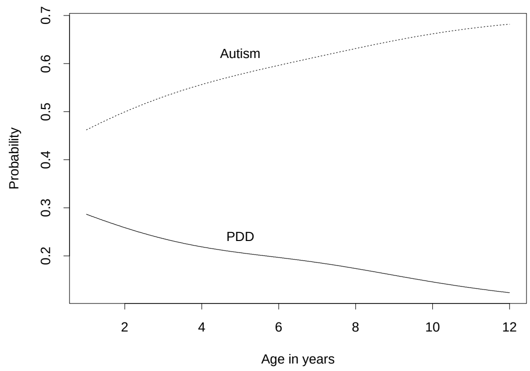
Marginal predicted probabilities of autism and pdd



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Example 3: Post-materialism study

A latent class model for rankings

In 1974/1975, over 2000 German respondents ranked four political goals according to their desirability:

1. Maintain order in the nation (ORDER)
2. Give people more say in decisions of the government (SAY)
3. Fight rising prices (PRICES)
4. Protect freedom of speech (FREEDOM)

Croon describes a latent class analysis of rankings.

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Model for Rankings

- Probability of a ranking of objects by subject j R_j can be written as

$$\Pr(R_j) \;=\; \frac{\exp(\nu_j^{r_j^1})}{\Sigma_{a=1}^{S}\exp(\nu_j^{r_j^a})} \times \frac{\exp(\nu_j^{r_j^2})}{\Sigma_{a=2}^{S}\exp(\nu_j^{r_j^a})} \times \cdots \times \frac{\exp(\nu_j^{r_j^S})}{\Sigma_{a=S-1}^{S}\exp(\nu_j^{r_j^a})}$$

  where r_j^a is the object given rank a.

- Each term represents the probability of choosing the object among the remaining objects.
- The latent class model is exploratory with

$$\nu_{jc}^s = e_c^s, \quad s = 1, 2, 3$$

$$\nu_{jc}^4 = 0$$

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Rankings Model Parameter Estimates

	one class	two classes	three classes
class 1			
probability	1	0.79	0.45
locations			
e_1^1 [ORDER]	1.16 (0.04)	1.94 (0.09)	1.84 (0.15)
e_1^2 [SAY]	0.21 (0.04)	0.21 (0.05)	0.17 (0.09)
e_1^3 [PRICES]	1.28 (0.04)	1.87 (0.09)	2.96 (0.31)
class 2			
probability		0.21	0.23
locations			
e_2^1 [ORDER]		-0.87 (0.09)	-0.76 (0.26)
e_2^2 [SAY]		0.44 (0.12)	0.56 (0.12)
e_2^3 [PRICES]		-0.21 (0.16)	-0.09 (0.19)
class 3			
probability			0.32
locations			
e_3^1 [ORDER]			3.14 (0.40)
e_3^2 [SAY]			0.21 (0.10)
e_3^3 [PRICES]			1.18 (0.16)
log-likelihood	-6427.05	-6311.69	-6281.36

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Data					Results for 3 class model				
						Posterior prob. (%)			
					pred.	class	1	2	3
Ranking				freq.	freq.	prior	0.45	0.23	0.32
1	2	3	4	137	126		10	10	**80**
1	2	4	3	29	46		1	31	**67**
1	3	2	4	309	315		36	3	**61**
1	3	4	2	255	257		37	2	**61**
1	4	2	3	52	40		2	26	**73**
1	4	3	2	93	93		11	6	**83**
2	1	3	4	48	50		20	**40**	**40**
2	1	4	3	23	29		2	**77**	22
2	3	1	4	61	57		**48**	46	6
2	3	4	1	55	61		7	**93**	0
2	4	1	3	33	32		1	**96**	3
2	4	3	1	59	61		2	**98**	0
3	1	2	4	330	339		**85**	3	12
3	1	4	2	294	281		**86**	2	12
3	2	1	4	117	109		**79**	18	4
3	2	4	1	69	56		25	**75**	0
3	4	1	2	70	81		**87**	9	4
3	4	2	1	34	41		32	**67**	0
4	1	2	3	21	18		3	**66**	32
4	1	3	2	30	30		27	21	**51**
4	2	1	3	29	25		2	**94**	4
4	2	3	1	52	47		3	**97**	0
4	3	1	2	35	33		**68**	23	9
4	3	2	1	27	33		13	**87**	0

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`gllamm` continues to develop.

Documentation, examples and the stata code are freely available from

http://www.iop.kcl.ac.uk/IoP/
Departments/BioComp/programs/gllamm.html