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Multilevel item response and structural equation modeling

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Outline

- Extending item response models
  - Two parameter logistic item response model
  - Including observed covariates
  - Multidimensional item response model
  - Multilevel item response model
  - Including latent covariates
- Generalized Linear Latent And Mixed Models (GLLAMM)
  - Response model
  - Structural model
- Estimation, prediction and simulation using Stata programs `gllamm`, `gllapred`, `gllasim`
- Application: Attitudes to Abortion

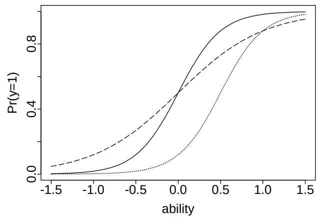
Two parameter logistic item response model

- Dichotomous response y_{ij} for item i and person j
- Two parameter logistic (2PL) model:

$$\text{logit}[\Pr(y_{ij}=1|\eta_j)] = \beta_i + \eta_j\lambda_i$$

$$\lambda_1 = 1$$

$$\eta_j \sim N(0, \tau^2)$$



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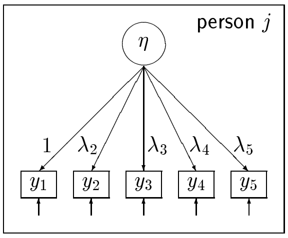
- η_j is the 'ability' of person j (factor)
- $-\beta_i/\lambda_i$ is the 'difficulty' of item i (β_i is an intercept)
- λ_i is the 'discrimination parameter' for item i (factor loading)



Including covariates

- Predictors of ability: w_j affects responses indirectly via η_j (MIMIC model)

$$\text{logit}[\Pr(y_{ij}=1|\eta_j)] = \beta_i + \eta_j\lambda_i, \quad \lambda_1 = 1$$

$$\eta_j = \gamma w_j + \zeta_j$$

$$\zeta_j \sim N(0, \tau^2)$$

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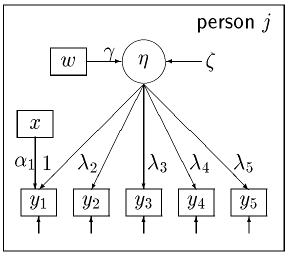
- Differential item functioning: x_j affects responses directly

$$\text{logit}[\Pr(y_{ij}=1|\eta_j, x_j)] = \beta_i + \eta_j\lambda_i + \alpha_i x_j, \quad \lambda_1 = 1$$



Multidimensional item response model

- Different types of ability (e.g. verbal, quantitative):

$$\text{logit}[\Pr(y_{ij}=1|\boldsymbol{\eta}_j)] = \beta_i + \eta_{1j}\lambda_{i1} + \eta_{2j}\lambda_{i2} + \eta_{3j}\lambda_{i3},$$

$$\lambda_{11}=\lambda_{42}=\lambda_{73}=1$$

- Confirmatory model:

$$\lambda_{41}=\lambda_{61}=\lambda_{71}=\lambda_{81}=\lambda_{91}=0$$
$$\lambda_{12}=\lambda_{22}=\lambda_{32}=\lambda_{72}=\lambda_{82}=\lambda_{92}=0$$
$$\lambda_{13}=\lambda_{23}=\lambda_{33}=\lambda_{43}=\lambda_{53}=\lambda_{63}=0$$

- Correlated abilities:

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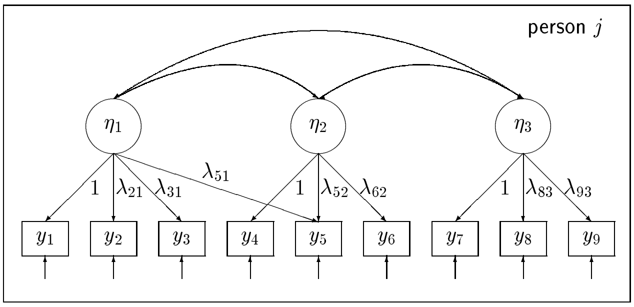
$$\begin{bmatrix} \eta_{1j} \\ \eta_{2j} \\ \eta_{3j} \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \tau_1^2 & & \\ \tau_{12} & \tau_2^2 & \\ \tau_{13} & \tau_{23} & \tau_3^2 \end{bmatrix} \right)$$



Multilevel item response model

- Students j (level 2) nested in schools k (level 3)
  (items i are at level 1)
- Predictors of ability: observed heterogeneity
  - Student-level predictor w_{1jk}, e.g. socio-economic status
  - School-level predictor w_{2k}, e.g. student-teacher ratio
- Omitted predictors: unobserved heterogeneity
  - Student-level residual $\zeta_{jk}^{(2)}$
  - School-level residual $\zeta_k^{(3)}$

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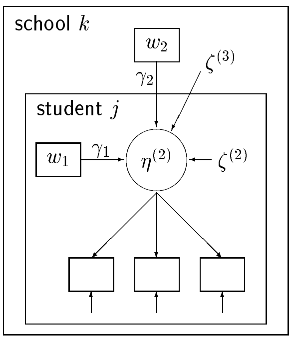
$$\eta_{jk}^{(2)} = \gamma_1 w_{1jk} + \gamma_2 w_{2k} + \zeta_k^{(3)} + \zeta_{jk}^{(2)}$$



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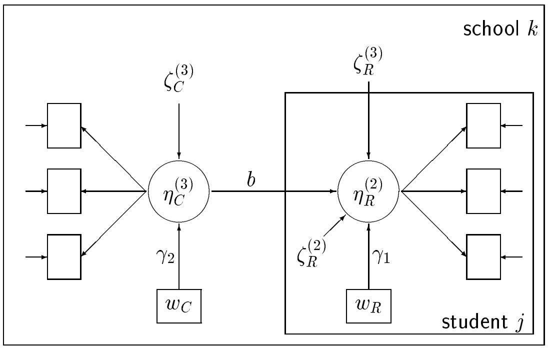
Including latent covariates

- Multilevel structural equation model
- Student ability $\eta^{(2)}_{Rjk}$
- Latent covariate at the school level $\eta^{(3)}_{Ck}$
  e.g. principal's attitude

$$\eta^{(2)}_{Rjk} = \gamma_1 w_{Rjk} + b\eta^{(3)}_{Ck} + \zeta^{(3)}_{Rk} + \zeta^{(2)}_{Rjk}$$

$$\eta^{(3)}_{ck} = \gamma_2 w_{Ck} + \zeta^{(3)}_{Ck}$$



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GLLAMM: Response model

The response model is a generalized linear model conditional on the latent variables

- 'Linear' predictor ν
  - for the 2PL model:
    $\nu_{ij} \;=\; \beta_i + \eta_j\lambda_i$
- Link function $g(\cdot)$, $\quad g(\mathrm{E}[y|\nu]) = \nu$
  - for the 2PL model:
    $\mathrm{logit}(\mathrm{E}[y_{ij}|\nu_{ij}]) \;=\; \nu_{ij}, \quad \mathrm{E}[y_{ij}|\nu_{ij}] \;=\; \Pr(y_{ij}=1|\nu_{ij})$
- Conditional distribution $f(y|\nu)$ from the exponential family
  - for the 2PL model:
    independent Bernouilli conditional on latent variable (ability)
    $\Longrightarrow$ conditional independence (or local independence)

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GLLAMM: Form of 'linear' predictor

- Multilevel data with L levels:
  units i (level 1) nested in clusters j (level 2), etc. up to level L
  - for 2PL model, $L = 2$:
    items i (level 1) nested in persons j (level 2)
- M_l latent variables at level $l = 1, \cdots, L$
  - for 2PL model, $M_2 = 1$
- Model for vector of linear predictors for level 2 unit j

$$\boldsymbol{\nu}_j = \mathbf{X}_j\boldsymbol{\beta} + \sum_{l=2}^{L}\sum_{m=1}^{M_l} \eta_m^{(l)} \mathbf{Z}_{mj}^{(l)} \boldsymbol{\lambda}_m^{(l)}$$

  - for 2PL model:

$$\underbrace{\begin{bmatrix} \nu_{1j} \\ \nu_{2j} \\ \nu_{3j} \end{bmatrix}}_{\boldsymbol{\nu}_j} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{X}_j} \underbrace{\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}}_{\boldsymbol{\beta}} + \eta_{1j}^{(2)} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{Z}_{1j}^{(2)}} \underbrace{\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}}_{\boldsymbol{\lambda}_1^{(2)}} = \begin{bmatrix} \beta_1 + \eta_{1j}^{(2)}\lambda_1 \\ \beta_2 + \eta_{1j}^{(2)}\lambda_2 \\ \beta_3 + \eta_{1j}^{(2)}\lambda_3 \end{bmatrix}$$

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Further examples of 'linear' predictor

$$\boldsymbol{\nu}_j = \mathbf{X}_j\boldsymbol{\beta} + \sum_{l=2}^{L}\sum_{m=1}^{M_l} \eta_m^{(l)} \mathbf{Z}_{mj}^{(l)} \boldsymbol{\lambda}_m^{(l)}$$

- Differential item functioning

$$\underbrace{\begin{bmatrix} \nu_{1j} \\ \nu_{2j} \\ \nu_{3j} \end{bmatrix}}_{\boldsymbol{\nu}_j} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & x_j \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\mathbf{X}_j} \underbrace{\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \alpha_1 \end{bmatrix}}_{\boldsymbol{\beta}} + \eta_{1j}^{(2)} \mathbf{Z}_{1j}^{(2)} \boldsymbol{\lambda}_1^{(2)} = \begin{bmatrix} \beta_1 + \eta_{1j}^{(2)}\lambda_1 + \alpha_1 x_j \\ \beta_2 + \eta_{1j}^{(2)}\lambda_2 \\ \beta_3 + \eta_{1j}^{(2)}\lambda_3 \end{bmatrix}$$

- Response model for multilevel structural equation model

$$\underbrace{\begin{bmatrix} \nu_{R1j} \\ \nu_{R2j} \\ \nu_{R3j} \\ \nu_{C1j} \\ \nu_{C2j} \\ \nu_{C3j} \end{bmatrix}}_{\boldsymbol{\nu}_j} = \mathbf{X}_j\boldsymbol{\beta} + \eta_{Rjk}^{(2)} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\mathbf{Z}_{1j}^{(2)}} \underbrace{\begin{bmatrix} \lambda_{R1} \\ \lambda_{R2} \\ \lambda_{R3} \end{bmatrix}}_{\boldsymbol{\lambda}_1^{(2)}} + \eta_{Ck}^{(3)} \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{Z}_{1j}^{(3)}} \underbrace{\begin{bmatrix} \lambda_{C1} \\ \lambda_{C2} \\ \lambda_{C3} \end{bmatrix}}_{\boldsymbol{\lambda}_1^{(3)}}$$

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Response processes

- Continuous
- Dichotomous
- Ordinal
  - Graded response model (cumulative) including models for thresholds or scale parameter
  - Partial credit model (adjacent odds)
  - Continuation ratio model
- Unordered categorical and rankings
- Counts
- Durations in continuous time
  - Proportional hazards model
  - Accelerated failure time model
- Durations in discrete time
  - Censored cumulative models
  - Continuation ratio model
  - Proportional hazards in continuous time
- Mixed responses

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Multinomial logit for categorical responses

- Categorical responses with S_i response alternatives for item i.
- If person j responds to item i with response s, $y_{sij} = 1$, $y_{tij} = 0$ for all $t \neq s$.
  - Example $S_1 = 3$, $S_2 = 2$:

$$\begin{bmatrix} y_{11j} \\ y_{21j} \\ y_{31j} \\ y_{12j} \\ y_{22j} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

- Specify GLLAMM model for vector of linear predictors

$$\boldsymbol{\nu}_j = (\nu_{1ij}, \cdots \nu_{S_1,ij}, \cdots, \nu_{1Ij} \cdots \nu_{S_I,Ij})'$$

- Response probability

$$\Pr(y_{sij} = 1 | \boldsymbol{\nu}_{ij}) = \frac{\exp(\nu_{sij})}{\sum_{t=1}^{S_i} \exp(\nu_{tij})}$$

- This seems to correspond to the Multidimensional Random Coefficient Multinomial Logit (MRCML) model (Adams, Wilson & Wang, 1997)

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GLLAMM: Structural model

Latent variables can be regressed on other latent and observed variables

$$\boldsymbol{\eta} = \boldsymbol{\Gamma}\mathbf{w} + \mathbf{B}\boldsymbol{\eta} + \boldsymbol{\zeta}$$

- $\boldsymbol{\eta}$ is an M-dimensional vector of latent variables

$$\boldsymbol{\eta} = (\eta_1^{(2)}, \eta_2^{(2)}, \cdots, \eta_{M_2}^{(2)}, \cdots, \eta_1^{(l)}, \cdots, \eta_{M_l}^{(l)}, \cdots, \eta_{M_L}^{(L)})'$$

- $\boldsymbol{\Gamma}$ is an $M \times p$ matrix of regression coefficients
- $\mathbf{w}$ is a p dimensional vector of explanatory variables
- $\mathbf{B}$ is an upper diagonal $M \times M$ matrix of regression coefficients
  - cannot regress higher level latent variable on lower level latent variable
  - relations among latent variables are recursive
- $\boldsymbol{\zeta}$ is an M-dimensional vector of errors/disturbances (each element $\boldsymbol{\zeta}$ varies at the same level as corresponding elements in $\boldsymbol{\eta}$).

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Examples of structural model

$$\boldsymbol{\eta} = \boldsymbol{\Gamma}\mathbf{w} + \mathbf{B}\boldsymbol{\eta} + \boldsymbol{\zeta}$$

- Multilevel item response model

$$\underbrace{\begin{bmatrix} \eta_{jk}^{(2)} \\ \eta_k^{(3)} \end{bmatrix}}_{\boldsymbol{\eta}} = \underbrace{\begin{bmatrix} \gamma_1 & \gamma_2 \\ 0 & 0 \end{bmatrix}}_{\boldsymbol{\Gamma}} \underbrace{\begin{bmatrix} w_{1jk} \\ w_{2k} \end{bmatrix}}_{\mathbf{w}} + \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} \eta_{jk}^{(2)} \\ \eta_k^{(3)} \end{bmatrix}}_{\boldsymbol{\eta}} + \underbrace{\begin{bmatrix} \zeta_{jk}^{(2)} \\ \zeta_k^{(3)} \end{bmatrix}}_{\boldsymbol{\zeta}}$$

$$\eta_k^{(3)} = \zeta_k^{(3)}$$

$$\eta_{jk}^{(2)} = \gamma_1 w_{1jk} + \gamma_2 w_{2k} + \underbrace{\zeta_k^{(3)}}_{\eta_k^{(3)}} + \zeta_{jk}^{(2)}$$

- Multilevel structural equation model

$$\underbrace{\begin{bmatrix} \eta_{Rjk}^{(2)} \\ \eta_{Ck}^{(3)} \\ \eta_{Rk}^{(3)} \end{bmatrix}}_{\boldsymbol{\eta}} = \underbrace{\begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \\ 0 & 0 \end{bmatrix}}_{\boldsymbol{\Gamma}} \underbrace{\begin{bmatrix} w_{Rjk} \\ w_{Ck} \end{bmatrix}}_{\mathbf{w}} + \underbrace{\begin{bmatrix} 0 & b & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} \eta_{Rjk}^{(2)} \\ \eta_{Ck}^{(3)} \\ \eta_{Rk}^{(3)} \end{bmatrix}}_{\boldsymbol{\eta}} + \underbrace{\begin{bmatrix} \zeta_{Rjk}^{(2)} \\ \zeta_{Ck}^{(3)} \\ \zeta_{Rk}^{(3)} \end{bmatrix}}_{\boldsymbol{\zeta}}$$

$$\eta_{Rk}^{(3)} = \zeta_{Rk}^{(3)}$$

$$\eta_{Ck}^{(3)} = \gamma_2 w_{Ck} + \zeta_{Ck}^{(3)}$$

$$\eta_{Rjk}^{(2)} = \gamma_1 w_{Rjk} + b\eta_{Ck}^{(3)} + \underbrace{\zeta_{Rk}^{(3)}}_{\eta_{Rk}^{(3)}} + \zeta_{Rjk}^{(2)}$$

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Estimation using the Stata program gllamm

- To obtain the likelihood of GLLAMM's, the latent variables must be integrated out
  - Sequentially integrate over latent variables, starting with the lowest level using a recursive algorithm
  - Use Gauss-Hermite quadrature to replace multivariate integrals by nested sums
  - Improve approximation using adaptive quadrature which scales and translates quadrature locations to match the peak of the integrand
- Maximize likelihood using Newton-Raphson

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Adaptive quadrature: unidimensional case

- Ordinary quadrature:

$$\begin{aligned}\ell_j(\boldsymbol{\beta},\boldsymbol{\lambda},\tau) &= \int \phi(\eta_j;0,\tau^2)\prod_i f(y_{ij}|\eta_j;\boldsymbol{\beta},\boldsymbol{\lambda})\mathrm{d}\eta_j,\ [u_j=\eta_j/\tau] \\ &= \int \phi(u_j)\prod_i f(y_{ij}|\tau u_j)\mathrm{d}u_j \\ &\approx \sum_{r=1}^{R} W_r \prod_i f(y_{ij}|\tau A_r)\end{aligned}$$

  - W_r and A_r: quadrature weights and locations
- Adaptive quadrature:

$$\begin{aligned}\ell_j(\boldsymbol{\beta},\boldsymbol{\lambda},\tau) &= \int \phi(u_j;\mu_j,\sigma_j^2)\left[\frac{\phi(u_j)\prod_i f(y_{ij}|\tau u_j)}{\phi(u_j;\mu_j,\sigma_j^2)}\right]\mathrm{d}u_j,\ [v_j=\frac{u_j-\mu_j}{\sigma_j}] \\ &= \int \frac{\phi(v_j)}{\sigma_j}\left[\frac{\phi(\sigma_j v_j+\mu_j)\prod_i f(y_{ij}|\tau(\sigma_j v_j+\mu_j))}{\frac{1}{\sqrt{2\pi}\sigma_j}\exp(-v_j^2/2)}\right]\sigma_j\mathrm{d}v_j \\ &\approx \sum_{r=1}^{R} W_r\left[\frac{\phi(\sigma_j A_r+\mu_j)\prod_i f(y_{ij}|\tau(\sigma_j A_r+\mu_j))}{\frac{1}{\sqrt{2\pi}\sigma_j}\exp(-A_r^2/2)}\right] \\ &= \sum_{r=1}^{R} \omega_{jr}\prod_i f(y_{ij}|\tau\alpha_{jr})\end{aligned}$$

  - $\phi(u_j;\mu_j,\sigma_j^2)$: normal density approximating posterior (approximately proportional to the numerator in [ ])
  - $\alpha_{jr}=\sigma_j A_r+\mu_j$
  - $\omega_{jr}=\sqrt{2\pi}\sigma_j\exp(A_r^2/2)\phi(\sigma_j A_r+\mu_j)W_r$

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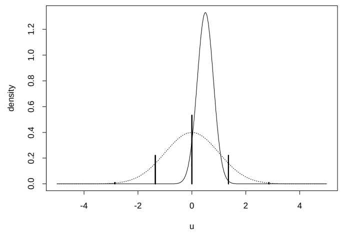
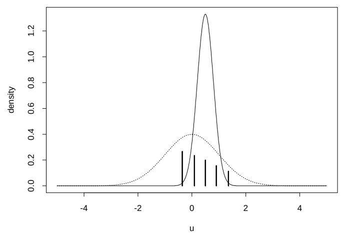
Adpative quadrature places quadrature locations under peak of integrand

Quadrature



Adaptive quadrature



Prior (dotted curve) and posterior (solid curve) densities

(Integrand is proportional to posterior)

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Prediction using `gllapred`

- Use empirical Bayes (EB) prediction to derive scores
  - Posterior mean for person j

$$\mathrm{E}[\eta_j|\mathbf{y}_j] = \frac{\int \eta_j \phi(\eta_j; 0, \hat{\tau}^2) \prod_i f(y_{ij}|\eta_j; \widehat{\boldsymbol{\beta}}, \widehat{\boldsymbol{\lambda}}) \mathrm{d}\eta_j}{\ell_j(\widehat{\boldsymbol{\beta}}, \widehat{\boldsymbol{\lambda}}, \hat{\tau})}$$

  - Posterior variance for person j

$$\mathrm{var}[\eta_j|\mathbf{y}_j] = \frac{\int \eta_j^2 \phi(\eta_j; 0, \hat{\tau}^2) \prod_i f(y_{ij}|\eta_j; \widehat{\boldsymbol{\beta}}, \widehat{\boldsymbol{\lambda}}) \mathrm{d}\eta_j}{\ell_j(\widehat{\boldsymbol{\beta}}, \widehat{\boldsymbol{\lambda}}, \hat{\tau})} - \mathrm{E}[\eta_j|\mathbf{y}_j]^2$$

  - The integrals are evaluated using adaptive quadrature

Simulation using `gllasim`

Simulate the abilities and responses for a given model

- Model diagnostics
  - Derive sampling distribution of measures of lack of fit
  - Compare diagnostic plot for real data with equivalent plots for simulated data
- Model interpretation
- Investigate (and correct) bias of estimators
- Investigate effects of model misspecification
- Power calculations

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Example syntax for 2PL

- Data:

person	item	i1	i2	i2	y
1	1	1	0	0	1
1	2	0	1	0	1
1	3	0	0	1	0
2	1	1	0	0	1
2	2	0	1	0	0
2	3	0	0	1	1

- Estimate model:

```
eq discrim: i1 i2 i3
gllamm y i1 i2 i3, i(person) eqs(discrim) /*
       */ link(logit) family(binom) adapt
```

- Obtain posterior means and standard deviations:

```
gllapred score, u
```

- Simulate abililties and responses:

```
gllasim abil, u
gllasim response
```

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Application: Attitudes to Abortion

- British Social Attitudes Survey Panel 1983-1986
- Respondents were asked whether or not abortion should be allowed by law if:

[wom] The woman decides on her own she does not wish to have the child

[cou] The couple agree that they do not wish to have the child

[mar] The woman is not married and does not wish to marry the man

[fin] The couple cannot afford any more children

[def] There is a strong chance of a defect in the baby

[ris] The woman's health is seriously endangered by the pregnancy

[rap] The woman became pregnant as a result of rape

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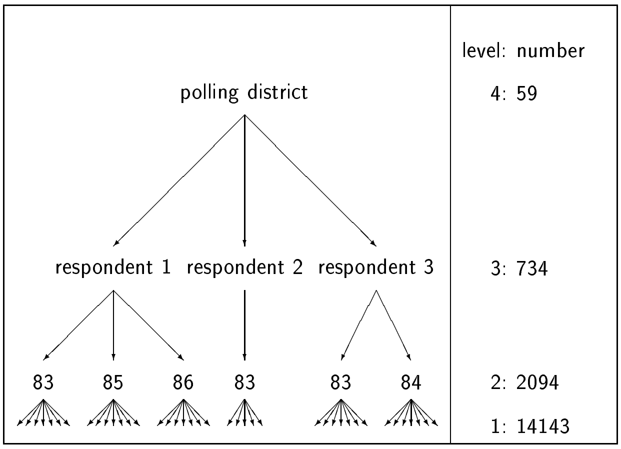
Data structure



- Unit non-response was common:

Number of panel waves participated in:	4	3	2	1
Percentage of respondents:	49	12	13	25

- Item non-response occurrred in only 7% of interviews

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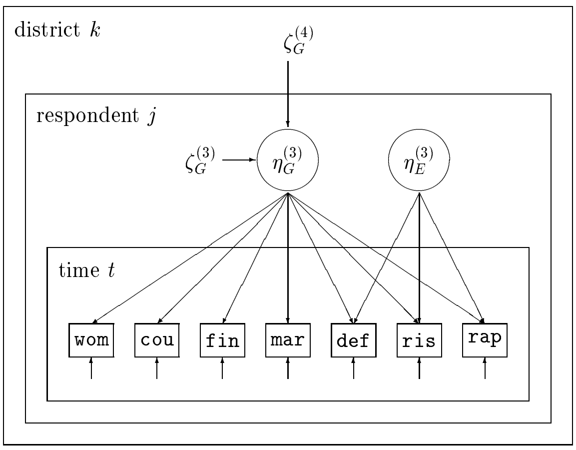
Models

- Only general attitude factor $\eta_G^{(3)}$
- Add extreme circumstance factor $\eta_E^{(3)}$, $X^2(3) = 207.7$
- (Add unique factors for respondents, $X^2(7) = 12.6$)
- Add district-level residual $\zeta_G^{(4)}$ for general attitude factor, $X^2(1) = 8.2$
- (Add district-level residual $\zeta_E^{(4)}$ for extreme circumstance factor, $X^2(1) = 3.2$)

Final Model



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FIXED PART

Intercepts:		
[wom]	-0.83 (0.14)	
[cou]	-0.17 (0.15)	
[mar]	-0.28 (0.16)	
[fin]	-0.01 (0.14)	
[def]	3.79 (0.27)	
[ris]	5.90 (0.56)	
[rap]	4.82 (0.39)	
RANDOM PART: Respondent level		
Factor loadings:	General	Extreme
[wom]	1	0
[cou]	1.13 (0.08)	0
[mar]	1.21 (0.09)	0
[fin]	1.01 (0.08)	0
[def]	0.78 (0.09)	1
[ris]	0.73 (0.13)	1.53 (0.26)
[rap]	0.72 (0.11)	1.23 (0.21)
Common factor variance:	5.22 (0.67)	3.30 (0.80)
RANDOM PART: District level		
Common factor variance:	0.36 (0.17)	0

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Some links and references

- The gllamm programs and manual can be downloaded from www.iop.kcl.ac.uk/iop/departments/biocomp/programs/gllamm.html

- Rabe-Hesketh, S., Skrondal, A. & Pickles, A. (2002). Generalized multilevel structural equation modelling. *Psychometrika*, conditionally accepted.

- Rabe-Hesketh, S., Skrondal, A. & Pickles, A. (2002). Reliable estimation of generalized linear mixed models using adaptive quadrature. *The Stata Journal, 2*, 1–21.

- Skrondal, A. & Rabe-Hesketh, S. (2002). Multilevel logistic regression for polytomous data and rankings. *Psychometrika*, in press.

- Skrondal, A. & Rabe-Hesketh, S. (2003). *Generalized latent variable modeling: Multilevel, longitudinal and structural equation models.* Boca Raton, FL: Chapman & Hall/ CRC.

If you would like a copy of any of the papers, email me: spaksrh@iop.kcl.ac.uk